Analysis of Fractional Systems using Haar Wavelet

Abdul Khader Valli T, Monica Mittal

Abstract: **Wavelets are relatively new tool and have quite been thriving domain in mathematical research. Numerical solutions of differential and integral equations require development of accurate and fast algorithms based on wavelets. This is more pertinent for those problems having localized solutions, both in position and scale. Haar wavelet offers a promising solution bases due to simple mathematical expressions and multi-resolution properties. In this paper, A Haar wavelet based method to solve partial differential equations (PDE) modeling fractional systems is presented. Operational approach is based on representing various integro-differential mathematical operations in terms of matrices. In this article, firstly introduction of Haar wavelet and different operational matrices used for the analysis of fractional systems are presented. A modified computational technique is explained to solve variety of partial differential equations modeling systems of fractional order. This method achieves the solutions by solving Sylvester equation using MATLAB. Demonstrations are provided with the help of two illustrative examples by suitable comparisons with exact solutions.**

Index Terms: *Fractional partial differential equations (FPDE), Haar wavelet, Fractional calculus, Operational matrices, Sylvester equation*

I. INTRODUCTION

Differential equations of fractional order are speculated from integer order derivatives, they are obtained by substituting derivatives of integer order by fractional ones. Compared with the derivatives of integer order, their benefits are ability to simulate dynamic systems and natural physical process more precisely [1]-[3]. For many researchers, fractional calculus has became the focus of interest. The recent investigation has unveiled its advantageous use in controlling and modeling many practical systems [4], [5]. For example, it is applied to continuum and statistical mechanics, fluid-dynamic traffic, economics, colored noise, signal processing, control theory [6]. The main difficulty of fractional differential equations is how to solve them, few techniques were suggested to solve differential equations of fractional order. Some of them are Generalized Differential Transform [7], [8] Variation Iteration technique [9], using Operational Matrix technique [10], Wavelet Method [11], [12], Finite Difference transform technique [13] and Adomian Decomposition Technique [14], [15], to name a few. But solutions are either difficult to be achieved or unachievable when the system is complex or of higher order. Therefore, there is a need for a computationally efficient and simple technique to achieve the solutions

This article focuses study on a variety of fractional partial differential equations (FPDE). Let partial differential equation of a system is described as

$$\frac{\partial^\alpha y}{\partial x^\alpha} + \frac{\partial^\beta y}{\partial t^\beta} = g(x,t) \tag{1}$$

And the subjected initial conditions are

$$y(0,t) = y(x,0) = 0$$

Where, $\frac{\partial^\alpha y(x,t)}{\partial x^\alpha}$ and $\frac{\partial^\beta y(x,t)}{\partial t^\beta}$ are Caputo sense derivative of fractional order, $y(x, t)$ is unknown function, g(x, t) is the known continuous function. The following are the some of the methods for solving FPDE. Podlubny [16] applied Laplace Transform technique for getting numerical solution of FPDE with constant coefficients. Odibat [17] used generalized differential transform technique to obtain numerical solution of FPDE. In this paper, modified numerical method based on operational matrices of Haar wavelet is presented for solving FPDE.

Haar wavelet is a group of orthogonal functions with piecewise fixed values, have been studied in recent years and applied as a useful tool for analysis, synthesis and other control problems. Because of their simplicity in formulations, sparsity and multi resolution Haar wavelet has definite advantages for solving problems involving derivatives and integrals in an efficient manner [18].

In the next section required definitions, notations and review of the fractional differential calculus are explained

II. DEFINITIONS

Definition 1

Definition of fractional differential operator in the sense of Caputo is given as follows [16]

$$D_*^\alpha y(t) = \begin{cases} \frac{d^r y(t)}{dt^r}, & \alpha = r \in N^+ \\ \frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}} d\tau, & 0 < r-1 < \alpha < r \end{cases} \tag{2}$$

Where, D_*^α is the differential operator of order α. $y(t)$ be the function defined for t > 0 and r is the positive natural number. The derivative of fractional order α in Caputo sense can also be conveyed as $D_*^\alpha y(t) = J^{r-\alpha} D^r y(t)$. Here D^r is the differential operator of order.

Definition 2

Fractional integral operator of Riemann–Liouville is defined as

$$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \tag{3}$$

$$J^0 y(t) = y(t) \tag{4}$$

And its properties are

$$J^\alpha J^\beta y(t) = J^{\alpha+\beta} y(t)$$

$$J^\beta y(t) = J^\beta J^\alpha y(t)$$

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\alpha+1)} t^{\alpha+\gamma}$$

Where, J^α is the integral operator of order α. $y(t)$ be the function defined for t > 0

III. HAAR WAVELET OPERATIONAL MATRIX

Haar functions are group of orthogonal functions. For $t \in [0,1]$ Haar functions are defined as, [21]

$$h_0(t) = \frac{1}{\sqrt{m}} \tag{5}$$

$$h_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{\frac{i}{2}}, & \frac{k-1}{2^j} \le t \le \frac{k-0.5}{2^j} \\ 2^{-\frac{i}{2}}, & \frac{k-0.5}{2^j} \le t < \frac{k}{2^j} \\ 0, & otherwise \end{cases} \tag{6}$$

$i = 0,1,2,\ldots,m-1$
$m = 2^{p+1}$ And p is the positive number. $i = 2^j + k - 1$ Where, j and k represents integer decompositions. Fig. 1 represents a set of Haar wavelet functions for m=4.

Haar functions integration can be extended into Haar series with a matrix p_m of size $m \times m$, it is known as integration operational matrix [17].
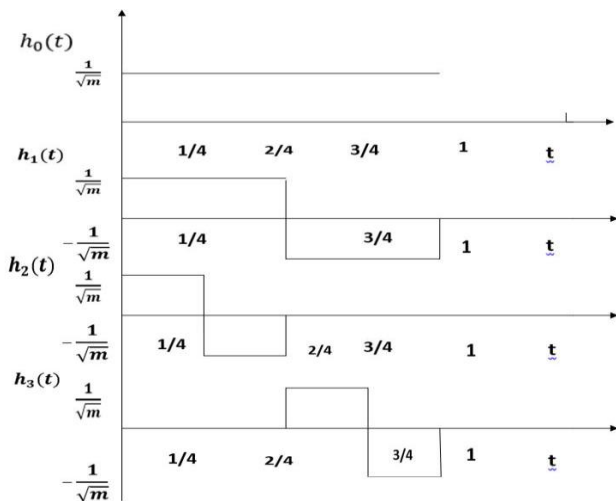

Fig. 1 Haar wavelets

The operational matrices of non-recursive formation are defined as [22]

$$p_m = h_m * Q_{bm} * h_m^{-1} \tag{7}$$

Where Q_{bm} is the block pulse integral operational and is defined

$$Q_{bm} = \frac{1}{m}\begin{bmatrix} 1/2 & 1 & \cdots & 1 \\ 0 & \cdots & \cdots & \vdots \\ \vdots & 0 & 1/2 & 1 \\ 0 & \cdots & 0 & 1/2 \end{bmatrix}$$

The block functions are defined in the interval $t \in [0,1)$ as

$$b_i(t) = \begin{cases} 1, & t \in \left[\frac{i}{m}, \frac{i+1}{m}\right] \\ 0, & otherwise \end{cases}; i = 0,1,\ldots m-1 \tag{8}$$

IV. SOLUTION OF THE FPDE

This section illustrates modified methodology to get numerical solution for FPDE. Consider a function with two variables i.e. $Y(x, t) = H^T(x).C.H(t)$,

Where, C is known as connection coefficient matrix. Integrating the function $Y(x,t)$ w.r.t. variable t Eq. (2) become

$$\left(\frac{\partial}{\partial t}\right)^\alpha Y = \left(\frac{\partial}{\partial t}\right)^\alpha (H^T(x).C.H(t)) = H^T(x).C.P_m^{-\alpha}.H$$

Similarly, integration w.r.t variable x Eq. (2) become

$$\left(\frac{\partial}{\partial x}\right)^\beta Y = \left(\frac{\partial}{\partial x}\right)^\beta (H^T(x).C.H(t)) = H^T(x).(P_m^{-\beta})^T C.H(t)$$

α and β are fractional orders combining both equations it become

$$\left(\frac{\partial}{\partial t}\right)^\alpha \left(\frac{\partial}{\partial x}\right)^\beta Y(x,t) = H^T(x).(P_m^{-\beta})^T C.P_m^{-\alpha}.H \tag{11}$$

Eq. (1) is considered for illustration. Function $Y(x, t)$ is approximated with help of Haar functions as

$$Y(x, t) = H^T(x).C.H(t).$$

Similarly the function $g(x,t)$ is also expressed as

$$g(x,t) = H^T(x).G.H(t) \tag{12}$$

Where,

$$G = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,m} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,m} \\ \vdots & \cdots & \cdots & \vdots \\ g_{m,1} & g_{m,2} & \cdots & g_{m,m} \end{bmatrix}$$

Now by substituting Eq. (9), Eq. (10) and Eq. (11) into Eq. (1) we get

$$H^T(x).(P_m^{-\beta})^T C.H(t) + H^T(x).C.P_m^{-\alpha}.H = H^T(x).G.H$$

Simplifying the above equation converts into

$$(P_m^{-\beta})^T C + C.P_m^{-\alpha}. = G \tag{14}$$

Eq. (14) is a Sylvester equation which can be

solved very easily by MATLAB.

V. NUMERICAL EXAMPLES

Example.1 [20]
Consider the non-homogeneous PDE

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 1, \qquad x, t \geq 0$$

Initial conditions are given as $y(x, 0) = y(0, t) = 0$
Exact solution of the given equation is given in [20]

$$y(x, t) = \begin{cases} t, & x \leq t \\ x, & x > t \end{cases}$$

The Fig. 2, Fig. 3 and Fig. 4 shows numerical solutions of m=8, 16and 32 respectively. In Table 1 we are comparing numerical solutions of varying m with the exact solution. Also from Fig. 6 it is clear that the error in numerical solution is reduces by increasing the resolution of the Haar wavelet.
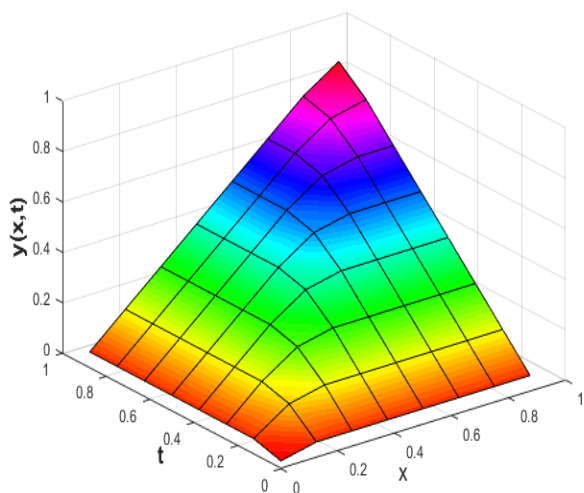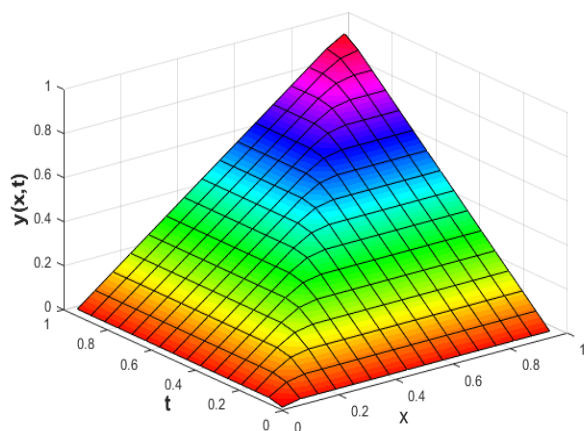

. Fig. 2 Numerical result of Example 1 with m=8
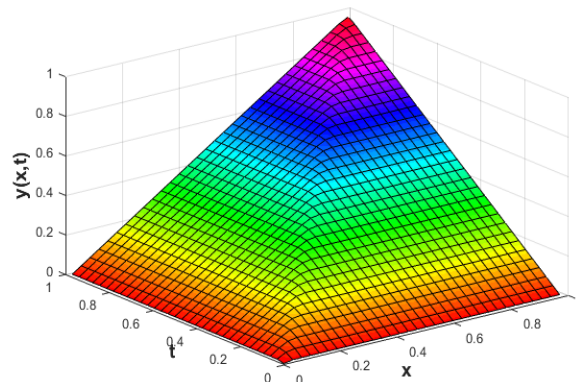

Fig. 3 Numerical result of Example 1 with m=16

Fig. 4 Numerical result of Example 1 with m=32
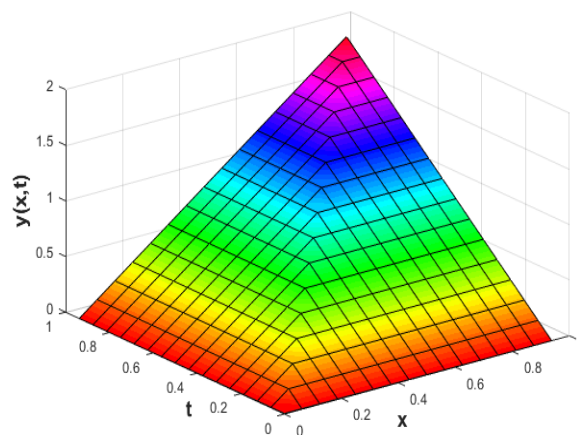

Fig. 5 Exact result of Example 1

Table 1. Comparison of numerical solution of m=8, 16 and 32 with Exact solution for Example 1

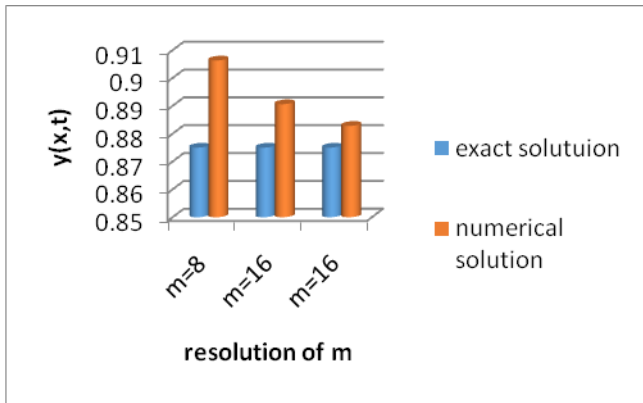| (x,t)	m=32	m=16	m=8	Exact results
(0,0)	0	0	0	0
(0.125,0.125)	0.1328	0.1406	0.1562	0.125
(0.250,0.250)	0.2578	0.2656	0.2813	0.250
(0.375,0.375)	0.3828	0.3906	0.4062	0.375
(0.500,0.500)	0.5078	0.5156	0.5313	0.500
(0.625,0.625)	0.6328	0.6406	0.6563	0.625
(0.750,0.750)	0.7578	0.7656	0.7813	0.750
(0.875,0.875)	0.8828	0.8906	0.9063	0.875

Fig. 6 Comparison of error of Haar and exact solutions at different m

Example.2 [20]

Given the following fractional FPDE

$$\frac{\partial^{1/2} y}{\partial t^{1/2}} + \frac{\partial^{1/2} y}{\partial x^{1/2}} = 1, \quad x, t \geq 0$$

Initial conditions are given as y(x, 0) = y(0, t)= 0 [20].By applying above method in MATLAB we get the numerical solutions of the problem are given in Fig. 6 and Fig. 7 respectively.
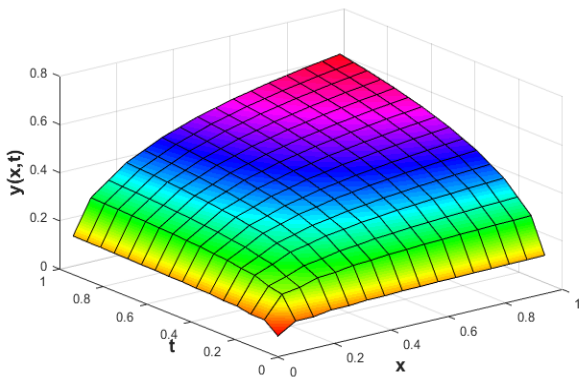

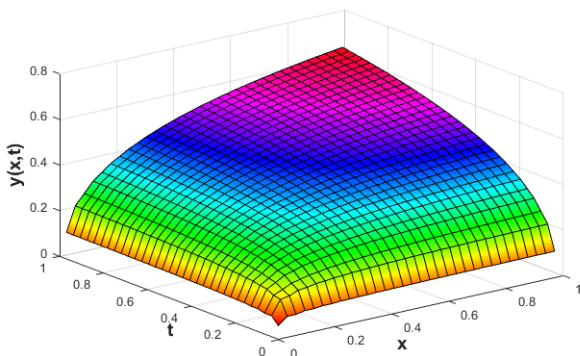Fig. 6 Numerical result of Exp 2 with m=16

Fig. 7 Numerical result of Example 2 with m=3

VI. CONCLUSION

This article uses the non recursive form of Haar integral operational matrix for solving partial differential equations of fractional order numerically. The highlight is that it converts problem into Sylvester equation. Advantages of this technique include (i) it is computer oriented and efficient; (ii) it is simple; (iii) the scope of application is wide. It is observed that by increasing m, numerical solution approaches to exact solution. The numerically unstable problem never occurs in this technique.

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