On A Class Non-Bazilević Function Involving Q-Differential Operator

K. R. Karthikeyan, K. Ramachandran, K. Srinivasan

***Abstract:** We define a q- differential operator which combines well-known Dziok-Srivastava operator and Sălăgean differential operator. Using this q-differential operator, we define a presumably new class of non-Bazilević function which has interesting subclasses of univalent functions as its special case and derive subordination and superordination results for the class in the unit disk. Further, interesting subordination conditions for starlikeness with respect k-symmetric points are obtained. Finally, we give relevant connections of our main results with former results obtained by various other authors.*

***Keywords:** q-calculus, univalent functions, starlike functions, convex function, Dziok-Srivastava operator, subordination, superordination.*

I. INTRODUCTION

We consider $\mathbb{A}$ is the set of all holomorphic functions in the form $F(z) = z + \sum_{m=2}^{\infty} c_m z^m$ (1.1) in the open disk with radius one $\Delta = \{ z: z \in \mathbb{C}: |z| < 1\}$. Also we let $\mathcal{S}$ to denote the subclass of $\mathbb{A}$ which are analytic and injective in Δ. We denote $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ are respectively starlike, convex, close-to-convex and quasi-convex in Δ of the familiar subclasses of $\mathbb{A}$. For detailed study on the development of various studies on univalent function theory, we refer to [5, 8].Let us consider $F_1(z_1)$ and $g_1(z_1)$ are holomorphic functions in Δ. Then we say $F_1(z_1)$ is subordinate to $g_1(z_1)$ in Δ, if there exists regular function $w_1(z_1)$ in Δ such as $|w_1(z_1)| < |z_1|$ and $F_1(z_1) = g_1(w_1(z_1))$, denoted by $F_1(z_1) \prec g_1(z_1)$. If $g_1(z_1)$ is one to one in Δ, then the subordination is equivalent to $F_1(0) = g_1(0)$ and $F_1(\Delta) \subset g_1(\Delta)$.Take k is a integer positive and $\varepsilon_k = \exp(2\pi i/k)$. For $F \in \mathbb{A}$, let $F_k(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-j} F(\varepsilon_k^j z)$. (1.2)

symmetric points.Similarly, define the $\mathcal{C}_s^{(k)}$ of convex functions with respect to k-symmetric points iff $Re\left(\frac{(zF'(z))'}{F_k'(z)}\right) > 0(z \in \Delta)$.(1.4) In the year 1908, Jackson [9] has introduced the Euler-Jackson q-difference operator

$D_q F(z) = \frac{F(z)-F(qz)}{z(1-q)} (z \in \Delta - \{0\};\ q \in \mathbb{C}\backslash\{0\})$, Here $\mathbb{C}$ represents the set of all complex numbers. As q tends to 1 is the derivative $\lim_{q\to 1} D_q F(z) = F'(z)$,provided the derivative exists.Forexample,

$D_q(z^\alpha) = \frac{z^\alpha-(qz)^\alpha}{z(1-q)} = [\alpha]_q z^{\alpha-1}, \alpha \in \mathbb{C}$,where $[m]_q = \sum_{k=1}^{m} q^{k-1}, [0]_q = 0,\ q in \mathbb{C}$. If $F(z)$ is in (1.1),then a easy computation gives

$D_q F(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, (z \in \Delta),$ (1.5)

and $D_q F(0) = F'(z) at 0,$ where $q \in (0,1)$.Heine developed the q-hypergeometric series as a generalization of thehypergeometric series ${}_2F_1[b, a;\ c|q, z] = \sum_{m=0}^{\infty} \frac{(b;q)_m (a;q)_m}{(q;q)_m (c;q)_m} z^m$(1.6)

where the q-shifted factorial is given by

$$(b;q)_m = \begin{cases} 1, & m = 0 \\ (1-b)(1-bq)\dots(1-bq^{m-1}), & m = 1,2,\dots, \end{cases}$$

and let us take $c \neq q^{-m}$ for $m = 0,1,2,\dots$. Generalization of Heine's series, we have ${}_r\Phi_s$ the basic hypergeometric seriesby

Revised Manuscript Received on May, 2019.

K.Ramachandran, Department of Mathematics, SRM University Ramapuram, Chennai-601206, Tamilnadu,India.

K. R. Karthikeyan , Department of Mathematics and Statistics, Caledonian College of Engineering, Muscat, Sultanate of Oman

K. Srinivasan, Department of athematics, Presidency College (Autonoumous), Chennai-600 005, Tamilnadu, India.

$l\phi_{m(})\ a_1,a_2,\ldots,a_l;b_1,b_2,\ldots,b_m;q,z$ =
∑undOvrn =
0∞(a1;q)n(a2;q)n …(al;q)n(q;q)n(b1;q)n …(bm;q)n[](−1)nq()on1n21+
m − lzn

(1.7) where $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $l > m+1$. In (1.6) and (1.7), the parameters $b_1, b_2, \ldots, b_m$ are denominators factors in the series are not equal to 0.For complex parameters $a_1, \ldots, a_l$ and $b_1, \ldots, b_m(\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-; \mathbb{Z}_0^- = 0,-1,-2,\ldots;\ j = 1,\ldots,m)$, we define the generalized q-hypergeometric function ${}_l\Psi_m(a_1,\ldots,a_l;b_1,\ldots,b_m;\ q,z)$ by

$${}_l\Psi_m(a_1,a_2,\ldots,a_l;b_1,b_2,\ldots,b_m;\ q,z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\ldots(a_l;q)_n}{(q;q)_n(b_1;q)_n\ldots(b_m;q)_n} z^n \quad (1.8)$$

$(l = m+1;\ l,m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in \Delta)$, where $\mathbb{N}$ be the set of positive integers. If $|q| < 1$, the series (1.8) converges absolutely for $|z| < 1$ and $l = m+1$.we refer to[7]. According to a function $\mathcal{G}_{l,m}(a_i,b_j;\ q,z)(i = 1,2,\ldots,l;\ j = 1,2,\ldots,m)$ is defined by

$$\mathcal{G}_{l,m}(a_i,b_j;\ q,z) := z\ \ {}_l\Psi_s(a_1,a_2,\ldots,a_l;b_1,b_2,\ldots,b_m;\ q,z)$$

(1.9) Define the operator $\mathcal{J}_\lambda^r(a_1,b_1;\ q,z)F : \Delta \longrightarrow \Delta$ by

$$\mathcal{J}_\lambda^0(a_1,b_1;\ q,z)F(z) = F(z) * \mathcal{G}_{l,m}(a_i,b_j;\ q,z)$$

$$\mathcal{J}_\lambda^1(a_1,b_1;\ q,z)F(z) = (1-\lambda)(F(z) * \mathcal{G}_{l,m}(a_i,b_j;\ q,z)) + \lambda\, zD_q(F(z) * \mathcal{G}_{l,m}(a_i,b_j;\ q,z))$$

(1.10)

$$\mathcal{J}_\lambda^r(a_1,b_1;\ q,z)F(z) = \mathcal{J}_\lambda^1(\mathcal{J}_\lambda^{r-1}(a_1,b_1;\ q,z)F(z))$$

(1.11) If $F \in \mathbb{A}_1$, then from (1.10) and (1.11) we deduce that

$$\mathcal{J}_\lambda^r(a_1,b_1;\ q,z)F = z + \sum_{n=2}^{\infty} \left[1-\lambda+[n]_q\lambda\right]^r \Upsilon_n c_n z^n,$$

(1.12) $(r \in N_0 = N \cup \{0\}\ and \lambda \geq 0)$, here $\Upsilon_s = \frac{(a_1;q)_{s-1}(a_2;q)_{s-1}\ldots(a_l;q)_{s-1}}{(q;q)_{s-1}(b_1;q)_{s-1}\ldots(b_m;q)_{s-1}}, (|q| < 1)$.

Remark 1 We note that the linear operator (1.12) is q-analogue of the operator defined by Selvaraj and Karthikeyan [16]. Here we list some special cases of the operator $\mathcal{J}_\lambda^m(a_1,b_1;\ q,z)F$.

1. Take $r = 0$, then the operator $\mathcal{J}_\lambda^0(\alpha_1,\beta_1)F(z)$ reduces to the q-analogue of Dziok- Srivastava operator [4].

2.For $a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0,-1,-2,\ldots,(i = 1,\ldots,l, j = 1,\ldots,m)$ and $q \to 1^-$, we have the operator defined by Selvaraj and Karthikeyan [16].

3. For $l = 2, m = 1; a_1 = b_1, a_2 = q$, and $\lambda = 1$, we get the q- analogue of the well known Sălăgean operator (see [11]). Fully in this paper, we $F_{\lambda,k}^m(a_1,b_1;\ q,z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-j}\left[\mathcal{J}_\lambda^r(a_1,b_1;\ q,\varepsilon_k^j z)F\right] = z + \cdots, (F \in \mathbb{A})$.

Let $\mathcal{P}$ is the class of holomorphic functions $h_1(z)$ with $h_1(0) = 1$, be convex and injective in Δ and in it $Re\{h_1(z)\} > 0, (z \in \Delta)$. Now we get this :

For $0 \leq \gamma_1 \leq 1$, a function $F(z) \in \mathbb{A}$ is in $\mathcal{N}_\lambda^r(a_1,b_1;\gamma_1;\ q;\ \phi)$ iff it satisfies the condition

$$D_q[\mathcal{J}_\lambda^r(a_1,b_1;\ q,z)]\left(\frac{z}{\mathcal{J}_\lambda^r(a_1,b_1;\ q,z)}\right)^{1+\gamma_1} \prec \phi(z), \quad \forall z \in \Delta. \quad (1.13)$$

For the choice (z)=1+z1-z, r=0, l=2, m=1; a_1=b_1, a_2=1, =1 and if we let $q \longrightarrow 1^-$. The class $\mathcal{N}_\lambda^r(a_1,b_1;\gamma_1;\ q;\ \phi)$ reduces to $\mathcal{N}(\gamma_1)(0 < \gamma_1 < 1)$ introduced recently by Obradović [15], he said this class to be non-Bazilević type.

Definition 1.1 [13] Let $\mathcal{Q}$ is the set of all holomorphic functions F_1 which are one to one on $\overline{\Delta} - E(F_1)$, here $E(F_1) = \{\zeta \in \partial\Delta: \lim_{z\to\eta} F_1(z) = \infty\}$, and is such that $F_1'(\zeta) \neq 0$ for $\eta \in \partial\Delta - E(F)$.

Lemma 1 [1](also see[13]) Let the function h is one to one in the open disc with radius one Δ and θ and ϕ are holomorphic in a domain $\mathbb{D}$ containing $k(\Delta)$ with $\phi(w) \neq 0$ when $w \in k(\Delta)$. set $Q(z) = zD_q(h(z))\phi(h(z))$, $k(z) = \theta(h(z)) + Q(z)$. Suppose that

1. Q is starlike and injective in Δ, and
2. $Re\left(\frac{zD_q k(z)}{Q(z)}\right) > 0\ for z \in \Delta$. If $\theta(p(z)) + zD_q p(z)\phi(p(z)) \prec \theta(h(z)) + zD_q h(z)\phi(h(z))$,

then $p(z) \prec h(z)$ and h is the best dominant.

Lemma 2 [3] Let the function h be one to one in the open disc has radius one Δ and ϑ and ϕ are holomorphic in a domain $\mathbb{D}$ having $h(\Delta)$. Suppose that

1. $Re\left(\frac{D_q[\vartheta(h(z_1))]}{\phi(h(z_1))}\right) > 0\ for z_1 \in \Delta$ and
2. $zD_q[h(z_1)]\phi(h(z_1))$ be starlike injective in Δ.

If $p \in \mathcal{H}[h(0),1] \cap Q$, with $p(\Delta) \subseteq \mathbb{D}$, and

$\vartheta(p(z_1)) + z_1 D_q p(z_1)\phi(p(z_1))$ is injective in Δ and $\vartheta(h(z_1)) + z_1 D_q h(z_1)\phi(h(z_1)) \prec \vartheta(p(z_1)) + z_1 p'(z_1)\phi(p(z_1))$,
then $h(z_1) \prec p(z_1)$ and h be the subordinant best.

II. CONDITIONS IN STARLIKENESS WITH RESPECT TO SYMMETRIC POINTS

Theorem 1 Consider the function $g_1(z)$ be convex injective in Δ and also let

$$Re\left\{\alpha_1\left(\frac{h(z_1)}{z_1 D_q h(z_1)}(h(z_1)-1)+1\right)+\beta\frac{h(z_1)}{zD_q h(z_1)}\right\}>0$$

(2.1) and

$$g_1(z) = \alpha_1 z D_q h(z_1) + \alpha_1 h^2(z_1) + (\beta-\alpha_1)h(z_1),$$

where $\alpha_1 > 0$, $\alpha_1+\beta>0$. If $F \in \mathbb{A}$ with F_, k^m(a_1,b_1; q, z)z 0 satisfies the condition

$$\alpha_1\left\{\frac{z_1^2 D_q^2(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} - \frac{z_1^2 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)D_q(F_{\lambda,k}^r(a_1,b_1;q,z_1))}{(F_{\lambda,k}^r(a_1,b_1;q,z_1))^2} + \frac{z_1^2[D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)]^2}{[F_{\lambda,k}^m(a_1,b_1;q,z_1)]^2}\right\} + \beta\frac{z_1 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} \prec h(z_1),$$

(2.2) then $\frac{zD_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)f)}{F_{\lambda,k}^r(a_1,b_1;q,z)} \prec g_1(z_1)$ and h is the best dominant.

Proof: Let p be

$$p(z_1) = \frac{z_1 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} (z_1 \in \Delta; z_1 \neq 0;\ F \in \mathbb{A}),$$

where $p(z_1) = 1 + p_1 z_1 + p_2 z_1^2 + \cdots \in \mathcal{P}$. then by a simple computation, we get

$$z_1 p'(z_1) = \frac{z_1 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} + \frac{z_1^2 D_q^2(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} - \frac{z_1^2 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)D_q(F_{\lambda,k}^r(a_1,b_1;q,z_1))}{(F_{\lambda,k}^r(a_1,b_1;q,z_1))^2}.$$

Thus by (2.2), we have

$$\alpha_1 z_1 D_q p(z_1) + \alpha_1 p^2(z_1) + (\beta-\alpha_1)p(z_1) \prec h(z_1).$$

(2.3) By setting

$$\theta(w) := \alpha_1 w^2 + (\beta-\alpha_1)w \quad and \quad \phi(w) := \alpha_1,$$

it is verified by θ is regular in $\mathbb{C}$, ϕ is regular in $\mathbb{C}$ with $\phi(0) \neq 0$ in the w-plane. Also, by letting

$Q(z_1) = z_1 D_q g_1(z_1)\phi(g_1(z_1)) = \alpha_1 z_1 D_q g_1(z_1)$ and $h(z_1) = \theta(g_1(z_1)) + Q(z_1) = \alpha_1(g_1(z_1))^2 + (\beta - \alpha_1)g_1(z_1) + \alpha_1 z_1 D_q g_1(z_1)$.

Since $g_1(z_1)$ is convex injective in Δ it implies that $Q(z_1)$ is starlike injective in Δ. Further, we have

$$Re\frac{z_1 D_q h(z_1)}{Q(z_1)} = Re\left\{\alpha_1\left(\frac{g_1(z_1)}{z_1 D_q g_1(z_1)}(g_1(z_1)-1)+1\right) + \beta\frac{g_1(z_1)}{z_1 D_q g_1(z_1)}\right\} > 0.$$

Corollary 1 If $f \in \mathbb{A}$ with F_, k^r(a_1,b_1; q, z_1)z_1 0 satisfies the condition

$$\alpha_1\left\{\frac{z_1^2 D_q^2(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} - \frac{z_1^2 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)D_q(F_{\lambda,k}^r(a_1,b_1;q,z_1))}{(F_{\lambda,k}^r(a_1,b_1;q,z_1))^2} + \frac{z_1^2[D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)]^2}{[F_{\lambda,k}^r(a_1,b_1;q,z_1)]^2}\right\} + \beta\frac{z_1 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} \prec h(z_1),$$

where

$$h(z_1) = \frac{\alpha_1(a-b)+(\beta-\alpha_1)[1+(a+bq)z_1+abqz_1^2]}{(1+bz_1)(1+bqz_1)} + \alpha_1\left(\frac{1+az_1}{1+bz_1}\right)^2,$$

$$-1 \le b < a \le 1 \quad and \quad \beta \ge 2\alpha_1^2\left(\frac{|b|}{1+|b|}+\frac{a-1}{1-b}\right)$$

then $\frac{z_1 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)f)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} \prec \frac{1+az_1}{1+bz_1}$.

Proof. We take g(z_1)=1+az_11+bz_1, in Theorem 1. Clearly $g(z_1)$ is convex injective in Δ. Hence this corollary proof follows from Theorem 1. If we let $q \to 1^-, r = 0, l = 2, m = 1; a_1 = b_1, a_2 = q, k = \lambda = 1$ in the Corollary 1, we have the below result.

Corollary 2 [18] If $F \in \mathbb{A}$ with F(z_1)z_1 0 satisfies the condition $\frac{z_1^2 F''(z_1)}{F(z_1)} + \beta\frac{z_1 F'(z_1)}{F(z_1)} \prec h(z_1)$,

where

$$h(z_1) = \frac{\alpha_1(a-b)+(\beta-\alpha_1)[1+(a+bq)z_1+abqz_1^2]}{(1+bz_1)(1+bqz_1)} + \alpha_1\left(\frac{1+az_1}{1+bz_1}\right)^2,$$

$$-1 \le b < a \le 1 \quad and \quad 2\left(\frac{|b|}{1+|b|}+\frac{1-a}{b-1}\right) \le \beta,$$

then $\frac{z_1 F'(z_1)}{F(z_1)} \prec \frac{1+az_1}{1+bz_1}$.

Corollary 3 If $f \in \mathbb{A}$ with F_, k^r(a_1,b_1; q, z_1)z_1 0, z_1 and $D = \mathbb{C}\backslash\left\{z_1 \in \mathbb{C}: Re\, z_1 \le -\frac{1}{2}, Im\, z_1 = 0\right\}$, then

$$\frac{z_1^2 D_q^2(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} - \frac{z_1^2 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)D_q(F_{\lambda,k}^r(a_1,b_1;q,z_1))}{(F_{\lambda,k}^r(a_1,b_1;q,z_1))^2} + \frac{z_1^2[D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)]^2}{[F_{\lambda,k}^r(a_1,b_1;q,z_1)]^2} + \frac{z_1 D_q(\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)F)}{F_{\lambda,k}^r(a_1,b_1;q,z_1)} \in D \Longrightarrow F \in \mathcal{S}_s^{(k)}.$$

Proof. If we assume $q \to 1^-, \alpha_1 = 1, \beta = 1$ and g(z)=1+z1-z in Theorem 1. It succeeds that $h(z)$ is convex in the point $u = -1/2$. Hence the proof.

put $k = 1$ in the above 3, we have the well-known outcome.

Corollary 4 [14] If $F \in \mathbb{A}$ with F(z)z 0, z and $D = \mathbb{C}\backslash\left\{z \in \mathbb{C}: Re\, z \le -\frac{1}{2}, Im\, z = 0\right\}$, then $\frac{z^2 F''(z)}{F(z)} + \frac{zF'(z)}{F(z)} \in D \quad \Longrightarrow Re\left(\frac{zF'(z)}{F(z)}\right) > 0$. If we let $\alpha = 1$, $q \longrightarrow 1^-, r = 0, l = 2, m = 1; a_1 = b_1, a_2 = q, \lambda = 1$ and $\mu = 1$ in the Corollary 5, then the result reduces to the assertion of the Theorem 1 follows by application of Lemma 1.

Corollary 5 If $F \in \mathbb{A}$ with F_k(z)z 0, z ,then $\left|\frac{zF'(z)}{F_k(z)}\left(1+\frac{F''(z)}{F'(z)}-\frac{zF_k'(z)}{F_k(z)}+\frac{zF'(z)}{F_k(z)}\right)-1\right|<2(z\in\Delta)$ implies |zF^'(z)F_k(z)-1|<1, for all $z\in\Delta$. If we let $k=1$ in the Corollary 5, we get the following interesting result.

Corollary 6 If $F\in\mathbb{A}$, then $\left|\frac{z_1F'(z_1)}{F(z_1)}\left(1+\frac{F''(z_1)}{F'(z_1)}\right)-1\right|<2(z_1\in\Delta),\Longrightarrow$ $\left|\frac{z_1F'(z_1)}{F(z_1)}-1\right|<1(z_1\in\Delta)$.

III. SUBORDINATION, SUPERORDINATION RESULTS FOR $\mathcal{N}_\lambda^m(a_1,b_1;\gamma;\ q;\ \phi)$

Theorem 2 Let δ is complex number except zero and let $k(z_1)$ is holomorphic and injective in Δ so $k(z_1)\neq 0,\forall z_1\in\Delta$. Suppose that z_1 D_qk(z_1)k(z_1)» is starlike univalent in Δ. Let $Re\{\frac{1}{\delta}\ k(z_1)+1+\frac{z_1D_q^2k(z_1)}{D_qk(z_1)}-\frac{z_1D_qk(z_1)}{k(z_1)}\}>0$ (3.1) and

$$\Psi_\lambda^m(a_1,b_1,\delta,f)(z_1):=D_q[\mathcal{J}_\lambda^r(a_1,b_1;\ q,z_1)](\frac{z_1}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})^{1+\gamma} \quad (3.2)$$
$$+\delta\left[\frac{z_1D_q^2[\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)]}{D_q[\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)]}+(1+\gamma)(1-\frac{z_1D_q[\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)]}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})\right]$$

If k satisfies the following subordination: $\Psi_\lambda^m(a_1,b_1,\delta,F)(z_1)\prec k(z_1)+\delta\frac{z_1D_qk(z_1)}{k(z_1)}$ then for $0\leq\gamma\leq 1$,

$$D_q[\mathcal{J}_\lambda^r(a_1,b_1;\ q,z_1)](\frac{z_1}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})^{1+\gamma}\prec k(z_1) \quad (3.3)$$

and $k(z_1)$ be the dominant best. Proof. Let p is defined by $p(z_1):=D_q[\mathcal{J}_\lambda^r(a_1,b_1;\ q,z_1)](\frac{z_1}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})^{1+\gamma}(z_1\in\Delta;z_1\neq 0;\ f\in\mathbb{A})$

Bysimplification,

$$\frac{\ln q}{q-1}\frac{z_1D_qp(z_1)}{p(z_1)}=\frac{\ln q}{q-1}\left[\frac{z_1D_q^2[\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)]}{D_q[\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)]}+(1+\gamma)(1-\frac{z_1D_q[\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)]}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})\right].$$

By taking $\theta(w):=w\quad and\quad \phi(w):=\frac{\delta}{w}$, it will be verified θ is holomorphic in $\mathbb{C}$, ϕ is holomorphic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0(w\in\mathbb{C}\backslash\{0\})$. Also, by letting $Q(z_1)=z_1D_q(k(z_1))\phi(k(z_1))=\delta\frac{z_1D_q(k(z_1))}{k(z_1)}$ and $h(z_1)=\theta(k(z_1))+Q(z_1)=k(z_1)+\delta\frac{z_1D_q(k(z_1))}{k(z_1)}$, so that $Q(z_1)$ is starlike injective in Δ and that $Re(\frac{z_1D_qh(z_1)}{Q(z_1)})=Re\{\frac{1}{\delta}\ k(z_1)+1+\frac{z_1D_q^2k(z_1)}{D_qk(z_1)}-\frac{z_1D_qk(z_1)}{k(z_1)}\}>0.$

This assertion (3.3) of the above Theorem 2 now follows by an application of Lemma 1.

For the choices k(z)=1+Az1+B z, -1 B <A 1 » and » k(z)=(1+z1-z)^,$0<\mu\leq 1$, in Theorem 2, we get the following results.

Corollary 7 Let δ is a complex number except zero and assume that (3.1) holds. If $f\in\mathbb{A}$ and $\Psi_\lambda^m(a_1,b_1,\delta,F)(z_1)(z_1)\prec\frac{1+Az_1}{1+Bz_1}+\delta\frac{(A-B)z_1}{(1+Az_1)(1+Bz_1)}$ where $\Psi_\lambda^m(a_1,b_1,\delta,F)(z_1)$ is in (3.2), then $\gamma\in[0,1]$, $D_q[\mathcal{J}_\lambda^r(a_1,b_1;\ q,z_1)](\frac{z_1}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})^{1+\gamma}\prec\frac{1+Az_1}{1+Bz_1}$ and 1+Az_11+B z_1» is the dominant best.

Corollary 8 Assume δ is a complex number other than zero and let it (3.1) holds. If $f\in\mathbb{A}$ and $\Psi_\lambda^m(a_1,b_1,\delta,f)(z_1)(z_1)\prec(\frac{1+z_1}{1-z_1})^\mu+\frac{2\delta\mu z_1}{(1-z_1^2)}$, here $\Psi_\lambda^m(a_1,b_1,\delta,f)(z_1)$ be in (3.2), then $\gamma\in[0,1]$, $D_q[\mathcal{J}_\lambda^r(a_1,b_1;\ q,z_1)](\frac{z_1}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})^{1+\gamma}\prec(\frac{1+z_1}{1-z_1})^\mu$ and (1+z_11-z_1)^» is the dominant best.

Next, by appealing to Lemma 2, we prove the following:

Theorem 3 consider δ is a complex number leaving zero and let k be analytic and injective in Δ so that $k(z)\neq 0$ and zD_qk^'(z)k(z) be starlike and $1-1$ in Δ.

Further,we take $Re[\frac{k(z)}{\delta}]>0.$ (3.4)

If $f\in\mathbb{A}$, $0\neq D_q(\mathcal{J}_\lambda^r(a_1,b_1;\ q,z))(\frac{z}{\mathcal{J}_\lambda^r(a_1,b_1;q,z)})^{1+\gamma}\in\mathcal{H}[k(0),1]\cap Q$ and $\Psi_\lambda^m(a_1,b_1,\delta,f)(z)$ is injective in Δ, then $k(z)+\delta\frac{zk'(z)}{k(z)}\prec\Psi_\lambda^m(a_1,b_1,\delta,f)(z)$ implies

$$k(z)\prec D_q(\mathcal{J}_\lambda^r(a_1,b_1;\ q,z))(\frac{z}{\mathcal{J}_\lambda^r(a_1,b_1;q,z)})^{1+\gamma} \quad (3.5)$$

and k is the best subordinant where $\Psi_\lambda^m(a_1,b_1,\delta,f)(z)$ is in (3.2).

Proof. By taking $\vartheta(w):=w\quad and\quad \phi(w):=\frac{\delta}{w}$, it must be verified that ϑ is regular in $\mathbb{C}$, ϕ is regular in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0(w\in\mathbb{C}\backslash\{0\})$. By the statement of the Theorem 3, $zk'(z)\phi(k(z))$ is starlike (injective) function and $Re\frac{\vartheta'(k(z))}{\phi(k(z))}=Re[\frac{k(z)}{\delta}]>0.$

The assertion (3.5) of Theorem 3 follows by an application of Lemma 2.Joining Theorem 2 and Theorem 3, we have the below sandwich theorem.

Theorem 4 Let δ be a complex number not zero and let k_1 and k_2 be $1-1$, so that $k_1(z_1) \neq 0$ and $k_2(z_1) \neq 0, \forall z_1 \in \Delta$ with z_1k_1^'(z_1)k_1(z_1)»and» z_1k_2^'(z_1)k_2(z_1) being starlike univalent.
Suppose that k_1 satisfies (3.4) and k_2 satisfies (3.1). If $f \in \mathbb{A}$,
$D_q(\mathcal{J}_\lambda^r(a_1,b_1;\, q,z_1))(\frac{z_1}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})^{1+\gamma} \in \mathcal{H}[k(0),1] \cap Q, \quad and$
$\Psi_\lambda^m(a_1,b_1,\delta,f)(z_1)$ is univalent in Δ, then
$k_1(z_1) + \delta\frac{z_1k_1'(z_1)}{k_1(z_1)} \prec \Psi_\lambda^m(a_1,b_1,\delta,f)(z_1)(z_1) \prec k_2(z_1) + \delta\frac{z_1k_2'(z_1)}{k_2(z_1)}$
implies
$k_1(z_1) \prec D_q(\mathcal{J}_\lambda^r(a_1,b_1;\, q,z_1))(\frac{z_1}{\mathcal{J}_\lambda^r(a_1,b_1;q,z_1)})^{1+\gamma} \prec k_2(z_1)$
and k_1 and k_2 are respectively the best subordinant and the dominant.
When $\gamma = 1, m = 0, r = 2, s = 1, a_1 = b_1 a_2 = q$ and $q \to 1^-$, we have the following corollary.

Corollary 9 [17] Let δ be a complex number not zero and let k_1 and k_2 be $1-1$ in Δ so that $k_1(z_1) \neq 0$ and $k_2(z_1) \neq 0, (z_1 \in \Delta)$ with z_1k_1^'(z_1)k_1(z_1)» and z_1k_2^'(z_1)k_2(z_1) being starlike univalent. Suppose that k_1 satisfies (3.4) and k_2 satisfies (3.1). If $F \in \mathbb{A}$,
$\frac{z_1^2F'(z_1)}{\{F(z_1)\}^2} \in \mathcal{H}[k(0),1] \cap Q,$ and
$\Psi_\lambda F(z_1) := \frac{z_1^2F'(z_1)}{(F(z_1))^2} + \delta[\frac{z_1F''(z_1)}{F'(z_1)} + 2(1 - \frac{z_1F'(z_1)}{F(z_1)})]$
is univalent in Δ, then
$k_1(z_1) + \delta\frac{z_1k_1'(z_1)}{k_1(z_1)} \prec \Psi_\lambda^m(a_1,b_1,\delta,F)(z_1)(z_1) \prec k_2(z_1) + \delta\frac{z_1k_2'(z_1)}{k_2(z_1)}$
implies $k_1(z_1) \prec \frac{z_1^2F'(z_1)}{\{F(z_1)\}^2} \prec k_2(z_1)$ and k_1 and k_2 are respectively the best subordinant and the dominant.
Remark 2: Several results in [16] could be obtained as a special case of our results if we let $a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, \dots, (i = 1, \dots, r, j = 1, \dots, s)$
and $q \to 1^-$.

IV. CONCLUSION

This paper construct the new class of non-Bazilević function and gives its comparative results.

REFERENCES

1. H. Aldweby and M. Darus, Some subordination results on q-analogue of Ruscheweyh differential operator, Abstr. Appl. Anal. 2014, Art. ID 958563, 6 pp.
2. A.Aral, V. Gupta and R. P. Agarwal, Applications of q —calculus in operator theory, Springer, New York, 2013.
3. T. Bulboacă, A class of superordination-preserving integral operators, Indag. Math. (N.S.) 13 (2002), no. 3, 301–311.
4. M. Darus, A new look at q-hypergeometric functions, TWMS J. Appl. Eng. Math. 4 (2014), no. 1, 16–19.
5. P. L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York, 1983.
6. T. Ernst, A comprehensive treatment of q-calculus, Birkhäuser/Springer Basel AG, Basel, 2012.
7. G. Gasper and M. Rahman, Basic hypergeometric series, Encyclopedia of Mathematics and its Applications, 35, Cambridge University Press, Cambridge, 1990.
8. A.W. Goodman, Univalent functions. Vol. II, Mariner Publishing Co., Inc., Tampa, FL, 1983.
9. F. H. Jackson, On q —functions and a certain difference operator, Trans. Roy. Soc. Edin., 46 (1908),253–281.
10. F. H. Jackson, On q —definite integrals, Quarterly J. Pure Appl. Math., 41 (1910),193–203.
11. K. R. Karthikeyan, M. Ibrahim and S. Srinivasan, Fractional class of analytic functions defined using q-differential operator, Aust. J. Math. Anal. Appl. 15 (2018), no. 1, Art. 9, 15 pp.
12. S. S. Miller and P. T. Mocanu, Univalent solutions of Briot- Bouquet differential equations, J. Differential Equations 56 (1985), no. 3, 297–309.
13. S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Var. Theory Appl. 48 (2003), no. 10, 815–826.
14. P. T. Mocanu and G. Oros, Sufficient conditions for starlikeness. II, Studia Univ. Babeş-Bolyai Math. 43 (1998), no. 2, 49–53.
15. M. Obradović, A class of univalent functions, Hokkaido Math. J. 27 (1998), no. 2, 329–335.
16. C. Selvaraj and K. R. Karthikeyan, Differential sandwich theorems for certain subclasses of analytic functions, Math. Commun. 13 (2008), no. 2, 311–319.
17. T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, Differential sandwich theorems for some subclasses of analytic functions, Aust. J. Math. Anal. Appl. 3 (2006), no. 1, Art. 8, 11 pp. (electronic).
18. N. Xu and D. Yang, Some criteria for starlikeness and strongly starlikeness, Bull. Korean Math. Soc. 42 (2005), no. 3, 579–590.