Primeradicals in Ternary Semi Groups

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Abstract*: *In this paper we study the properties of prime radical of an ideal in a ternarysemigroup. We characterize different classes of ternarysemigroups by their properties of their radicals and nilpotent. We introduced and charaterize the notions of radical ideal generated by P in ternarysemigroups.

I. INTRODUCTION

The literature of ternary algebraic system was introduced by D.M.Lehmer in 1932. The notion of ternarysemigroups was known to S. Banach.

He showed by an example that a ternarysemigroup does not necessarily reduce to an ordinary semigroup. Bindu [2] developed the properties of prime and maximal ideals in ternarysemigroups.

Some significant results are given in [3,4].

II. MAIN RESULT

Definition 2.1: Let P be an ideal of a ternarysemigroup T. Then primeradical of P, symbolized by $\beta(P)$ is defined as

$\beta(P)$ ={ $\cap$ of all primeideals of T each of which contains P}.

Definition 2.2: Let T be a Ternarysemigroup and Q be an ideal of T. Then Q is called nilpotent ideal if $Q^{2n+1} = 0$ for some n∈Z and $n \geq 0$.

Theorem 2.3: For an ideal P of a ternary semigroup T. Then

(a) $P \subseteq \beta(P)$.
(b) If A is a primeideal of T then $P \subseteq A$ if and only if $\beta(P) \subseteq A$.
(c) $\beta(P) \subseteq \beta(Q)$ where Q is an ideal in T satisfying the condition $P \subseteq Q$.
(d) $\beta(P)$ is a semi primeideal of T.
(e) $\beta(P) = \beta(P^{2n+1})$; n∈Z and $n \geq 0$.
(f) Every nilpotent element of T is contained in $\beta(P)$
(g) $\beta(\beta(P)) = \beta(P)$.

Proof: By the definition of primeradical we can easily prove (a),(b), (c).

(d) We know that $\beta(P)$ is an ideal of T.. Let $R^3 \subseteq \beta(P)$; where R is an ideal of T.

Now $\beta(P) = \cap \{A_i/P \subseteq A_i, A_i$ is a primeideal in T}. So $R^3 \subseteq A_i$, $\forall A_i$. Then A_i is prime; $R \subseteq A_i$, $\forall A_i$. Therefore $R \subseteq \beta(P)$. Hence $\beta(P)$ is a semi primeideal of T.

(e) P is an ideal in T, $P^{2n+1} \subseteq P$; n ∈Z and $n \geq 0$. Thus by (c), $\beta(P^{2n+1}) \subseteq \beta(P)$.

Supoose $x \in \beta(P)$. Now $\beta(P) = \cap\{A_i/P \subseteq A_i, A_i$ is prime ideal in T}.

Then $x \in A_i \forall A_i$. Suppose we assume that $x \notin \beta(P^{2n+1})$. Then ∃ a primeideal $B \in T \ni B \supseteq P^{2n+1}$ and $x \notin B$. Since B is prime, $P^{2n+1} \subseteq B \Rightarrow P \subseteq B$ whence B is some A_i, Which is a contradiction. Hence $x \in \beta(P^{2n+1})$. Hence $\beta(P) = \beta(P^{2n+1})$.

(f) Let M be the nilpotentideal in T. Then $M^{2n+1} = \{0\}$ for some n∈Z and $n \geq 0$. Hence $M^{2n+1} \subseteq \beta(P)$. Further $M^{2n+1} \subseteq A_i$ $\forall$ $A_i \supseteq P$ and A_i is a primeideal. Then $M \subseteq A_i \forall A_i$. Therefore $M \subseteq \beta(P)$.

(g) By (a); $P \subseteq \beta(P)$. By (c) $\beta(P) \subseteq \beta(\beta(P))$. Let $x \in \beta(\beta(P))$ and $\{A_i\}_{i\in I}$ be the group of primeideals in $T \ni P \subseteq A_i$ $\forall i \in I$. By the definition $\beta(P) \subseteq A_i \forall i \in I$. Whence $\beta(\beta(P)) \subseteq A_i$. Thus $x \in A_i \forall i \in I$. Whence $x \in \beta(P)$. Hence $\beta(\beta(P)) = \beta(P)$.

Theorem 2.4: [4] Let P be an ideal in a ternarysemigroup T. Then $\beta(P)$ = {t ∈ T/ every m-system in T which contains t, has a nonempty intersection with P}.

Proposition 2.5: Let P be an ideal in a ternarysemigroup T. If $x \in \beta(P)$ then ∃ an integer $n \geq 0 \ni x^{2n+1} \in P$.

Proof: Let $x \in \beta(P)$. Then by theorem 2.4, every m- system in T containing x has a nonempty intersection with P. Consider $M = \{x^{2n+1}/n \in Z$ and $n \geq 0\}$. Then M is an m-system containing x. Therefore $M \cap P \neq \emptyset$. Then ∃ an integer $n \geq 0 \ni x^{2n+1} \in P$.

Proposition 2.6: Suppose T is a commutative ternarysemigroup and M is an m-system in T which contains x. Then ∃ an integer $n \geq 0 \ni x^{2n+1}ab \in M$ where $a,b \in T$.

Proof: We recall the definition of commutativity and m-system of T. Since $x \in M$, ∃ a_1,a_2,a_3,a_4 in $T \ni xa_1xa_2x \in M$ or $xa_1a_2xa_3a_4x \in M$ or

$xa_1a_2xa_3xa_4 \in M$ or $a_1xa_2xa_3a_4x \in M$. This will imply that $x(a_1xa_2)x \in M$ or T being commutative, $x^3a_1a_2 \in M$ or $x^3a_1a_2a_3a_4 \in M$.

Let $x^3a_1a_2 \in M$. Then ∃ $a_5,a_6,a_7,a_8 \in T \ni x^5a_1a_2a_5a_6 \in M$ or $x^5 a_1a_2a_5a_6a_7a_8 \in M$.

Let $x^3a_1a_2a_3a_4 \in M$. Then ∃ $b_1,b_2,b_3,b_4 \in T \ni x^5a_1a_2a_3a_4b_1b_2 \in M$ or $x^5a_1a_2a_3a_4b_1b_2b_3b_4 \in M$. Proceeding in this process we get for every integer $n \geq 0$, $x^{2n+1}ab \in M$ for some $a,b \in T$.

Proposition 2.7: Suppose P be an ideal in a commutative ternarysemigroup $T \ni x^{2n+1} \in P$, where $x \in T$, $n \in Z$ and $n \geq 0$. Then $x \in \beta(P)$.

Proof: Let M be any m-system in T and x∈T. Then by above Proposition 2.6, $x^{2n+1}ab \in M$, for some $a,b \in T$. As P is an ideal and $x^{2n+1} \in P$, $x^{2n+1}ab \in P$. So $M \cap P \neq \emptyset$. Therefore by Theorem 2.4, $x \in \beta(P)$.

By above Propositions 2.5 and 2.7 we prove the following theorem.

Theorem 2.8: Let T be a commutative ternarysemigroup and P be an ideal of T. Then $\beta(P) = \{x \in T/x^{2n+1} \in P$ for some $n \in Z^+$ and $n \geq 0\}$

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Definition 2.9: An ideal P in a ternarysemigroup T is called a prime radicalideal if β(P) = P.

Prime radicalideal simply called as a radicalideal.

Proposition 2.10: The consecutive conditions in an ideal P of a ternarysemigroup T are identical:

(a) β(P) = P

(b) $x^{2n+1} \in P \Rightarrow x \in P$, where $n \in Z$ and $n \geq 0$.

Proof: (a) ⇒ (b). Let $x^{2n+1} \in P$ then by above Proposition 2.7, $x \in \beta(P) = P \Rightarrow x \in P$.

(b) ⇒ (a). We have $P \subseteq \beta(P)$. Let $t \in \beta(P)$. Then by above Proposition 2.5 ∃ an integer $n \geq 0 \ni t^{2n+1} \in P$. Hence by (b) $t \in P$. Hence $\beta(P) \subseteq P$. Therefore β(P) = P.

Theorem 2.11: In a ternarysemigroup intersection of any set of radicalideals is a radicalideal.

Proof: Let T be a ternarysemigroup and $\{S_i / i \in \Lambda\}$ be any set of radicalideals in T. Then by above definition 2.9, $\beta(S_i) = S_i$. Now $\cap S_i \subseteq S_i \ \forall i \in \Lambda$. So by above proposition 2.3 (c), $\beta(\bigcap_{i\in\Lambda} S_i) \subseteq S_i \ \forall i \in \Lambda$. Therefore $\beta(\bigcap_{i\in\Lambda} S_i) \subseteq \bigcap_{i\in\Lambda} S_i \ \forall i \in A$. Again $\bigcap_{i\in\Lambda} S_i \subseteq \beta(\bigcap_{i\in\Lambda} S_i)$ (by proposition 2.3(a)). Therefore $\beta(\bigcap_{i\in\Lambda} S_i) = \bigcap_{i\in\Lambda} S_i \ \forall i \leq i \in \Lambda$. Hence $\bigcap_{i\in\Lambda} S_i$ is a radicalideal

Definition 2.12: Let T is a ternarysemigroup with a subsemigroup P and for an ideal I, A = I ∩P is an ideal. If there is some other ideal J ∈ T ∋ I ⊆ J and A = J ∩P then we say that I can be extended to an ideal in T which will also contracts to A.

Theorem 2.13: Suppose P be an *m*-system and N be an ideal of a ternarysemigroup T ∋ N ∩ P = ∅. Then ∃ a maximalideal M of T contained N ∋M ∩ P = ∅. Further, M is a primeideal of T.

Theorem 2.14: Let T be a commutative ternarysemigroup and P be a ternary subsemigroup of T. Let I be a radicalideal of T ∋ xyz ∈ I, x ∈ P; y,z ∈ T implies that either x ∈ I or y ∈ I or z ∈ I. Then A = I ∩ P is a primeideal in P. Moreover I can be declared as an intersection of primeideals each of which contradicts to A.

Proof & Results: Suppose x,y,z ∈ P ∋ xyz ∈ A. Then xyz ∈ I. Therefore by our assumption either x ∈ I or y ∈ I or z ∈ I. Thus either x ∈ A or y ∈ A or z ∈ A. So A becomes a primeideal. Put X = {J : J is a primeideal of T with J ⊇ I and J ∩ P = A}. Then I ⊆ X. Now we prove X ⊆ I. Let x ∉ I. Then the m-system $m = \{y\} \cup \{dy^{2n}/d \in P$ but $d \notin A$ and $n \in Z^+\}$ and m∩I=ø . Then by proposition 2.13 ∃ a maximalideal B ⊇ I with B ∩ M = φ which is again prime.

Then A ⊆ B∩P. Again for b ∈ B∩P, $by^2 \in B$; B is an ideal of T. It follows that $by^2 \notin M$. This is together with the definition of M and b ∈ P implies that b ∈ A. Therefore B ∩ P ⊆ A. Hence A = B ∩ P. Again y ∉ B as y ∈ M and M ∩ B = φ. Therefore y ∉ X and X ⊆ I. Hence I = X.

Definition 2.15: Let *T* be a ternarysemigroup and P ⊆ T for some P . Let {P} be the radicalideal generated by P and is defined as the intersection of all radicalideals of T such that each ideal contains P. Clearly {P} is the smallest radicalideal that contains P. We denote {P,x} as {P ∪{x}}.

Theorem 2.16: In a commutative ternarysemigroup T satisfying ascending chain condition on radicalideal can be expressed as the finite intersection of primeideals.

Proof: Let T be a commutative ternarysemigroup satisfying ascending chain condition on radicalideals. Put R is the set of all radicalideals which cannot be expressed as the finite intersection primeideals and R ≠ ∅. T satisfies ascending chain condition on radicalideals, R has a maximal element say I. Since I ∈ R and it cannot be expressed as the finite intersection of primeideals, i.e I is not prime.

Therefore ∃ x,y,z ∈ S ∋ xyz ∈ I but x ∉ I,y ∉ I,z ∉ I.

Then each of the radicalideals {I,x},{I,y},{I,z} ⊇ I. Therefore each of them can be expressed as the finite intersection of primeideals in T.

Now {I,x},{I,y},{I,z}⊆ I,{xyz}⊆ I. For any b ∈{I,x}∩{I,y}∩{I,z}, $b^3 \in I \Rightarrow b \in I$ as I is a radicalideal. So {I,x}∩{I,y};{I,z}⊆ I. Clearly I ⊆{I,x}∩ {I,y}∩{I,z}. Therefore I = {I,x}∩{I,y}∩{I,z}. Hence I can be expressed as the finite intersection of primeideals, which is a contradiction. Therefore R = ∅. Hence the proof.

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